PROMPTED AND UNPROMPTED TRANSITIONS BETWEEN REPRESENTATIONAL MODES IN CALCULUS

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This article argues for a shift in how researchers discuss and examine students' uses of representations during their calculus problem solving. An extension of Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis-framework to include physical modes of reasoning is proposed. An example that details how transitions between visual, analytic and physical reasoning inform students' problem solving in a calculus context is discussed.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking

Background

Cartesian graphs, analytic notation and real/imagined physical scenarios are ways of representing a function. These three modes of representation provide very different opportunities for reasoning about function and translating between these modes is non-trivial (Leinhardt, Zaslavsky, & Stein, 1990).

The extent to which function is tied to physical scenarios varies depending on the particular curriculum. However, often emphasized is the connection between modes. Several researchers have explored understandings of the connection between graphic and analytic modes through tasks that prompt students to translate between these modes (e.g. Kaput, 1987; Knuth 2000). Additionally, there is a growing literature on mathematization of real and imagined scenarios and interpretation of what graphs imply about the situations they describe (Gravemeijer, & Doorman, 1999; Nemirovsky, Tierney, & Wright, 1998). Achieving representational fluency is an important part of secondary school mathematics (Knuth, 2000). This type of fluency later plays an important role in calculus.

However, very little calculus education research has focused specifically on transitions between representations. Education Researchers and curriculum developers with strong ties to the education research community have, for years, stressed the importance of including multiple representations as part of a calculus course. One major product which emerged from the Calculus reform movement of the late 1980's and early 90's is Hughes-Hallet et. al's (1994) calculus text. A guiding principle which underpins this textbook's approach is "the 'Rule of Three,' which says that whenever possible, topics should be taught graphically and numerically, as well as analytically" (p. 121). To date this text still comprises 19% of the US calculus textbook market (Bressoud, 2011). In recent years the 'Rule of Three' has been appended to include physical/kinesthetic considerations and has become a 'Rule of Four' (Kung & Speer, in press).

Many researchers have echoed the cry to emphasize more than just the symbolic aspects of calculus. Zimmerman (1991) wrote that, "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136). However, in spite of the emphasis on multiple modes of representation, students' difficulties with implementing non-symbolic modes of reasoning have been a prevalent theme in the literature. For example, in a series of studies that explored multiple representations Presmeg and colleagues have emphasized the rareness of students who are comfortable with both analytic (verbal-logical) and graphical (visual pictorial) thinking

(Aspinwall, Shaw, & Presmeg, 1997; Presmeg, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2010). They refer to such students as harmonic thinkers. In general, these studies have concluded that students who prefer graphical modes of reasoning are rare and harmonic thinkers are even rarer. Some authors have attributed students' difficulties with and resistance to visual reasoning in calculus to the over-emphasis on symbolic representations in secondary grades (Haciomeroglu, Aspinwall, & Presmeg, 2010; Vinner, 1989).

Furthermore, discussion of multiple representations has been expanded beyond analytic and graphic modes. Zandieh's (2000) derivative framework includes verbal and kinesthetic reasoning in addition to analytic and graphical. However, the work on multiple representations in calculus has tended to focus on which representations are present in student thinking on particular tasks, rather than how transitions between representational modes contribute, whether productively or not, to student problem-solving in calculus. Given the amount of emphasis translation between representation modes has received in secondary school, it seems natural to extend the investigation of this phenomena into calculus education research.

Theoretical Perspective

In this section I introduce the Visualization/Analysis-model (VA-model) and suggest an extension of it that includes physical modes of reasoning. I argue that a more detailed look at transitions among different modes of reasoning is needed and I exemplify different types of transitions in calculus tasks.

The VA-Model

My perspective extends the VA-model (Zazkis, Dubinsky, & Dautermann, 1996), which views the development of visual and analytic modes of reasoning as complementary rather than disjoint processes (see Figure 1). The modes, which may start as wholly separate entities, build on one another as reasoning develops. As students progress, their ability to translate between these modes becomes more common and the connections between the modes become stronger. In other words, the model contends that a back-and-forth relationship between the modes of reasoning does not develop overnight; it develops over time, and as it does, the transition between modes becomes progressively more natural for students to make. Figure 1 illustrates this process via a path through successive levels of visualization and analysis in which the 'distance' between visualization and analysis decreases as the levels advance.

Although the VA-model was used in prior research, these studies tend to focus on classifying individual students as visual thinkers, analytic thinkers or harmonic thinkers (e.g. Haciomeroglu, Aspinwall, & Presmeg, 2010). I see this classification as inconsistent with the VA-model since the model contends that all students make transitions between modes during their mathematical development. In other words, the model contends, implicitly, that all students are harmonic thinkers.

The tendency to shy away from examining transitions between representation modes is also present in work that does not subscribe to the VA-model. For example, Zandeih's (2000) derivative framework classifies individual students in terms of whether or not they have expressed various modes of thinking. That is, rather than focusing on transitions between modes of thinking and how one mode informs another, students were classified in terms of the modes they expressed.

Focusing specifically on transitions between representations stays true to the VA-model. Additionally, it helps illuminate how harmonic thinking develops, even in students that rarely use certain modes of thinking.

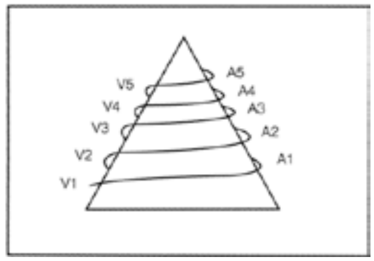

Figure 1: The VA-Model Diagram (from Zazkis et al. (1996))

The Expanded VA-Model

In line with Zandieh's work and the general trend in calculus education research to include real or imagined physical scenarios, my model adds representations that are based on real or imagined physical scenarios to the VA-model. This is consistent with a growing body of work that emphasizes the importance of physical contexts in students' understanding of calculus, such as the relationship between acceleration, velocity and position (Nemirovsky, Tierney, & Wright, 1998). In order to reflect this change the model will henceforth be referred to as the VAP-model. Mathematics is often motivated by connections to physical scenarios. So I see this addition to the model, which was originally not developed for calculus, as applicable to other areas of mathematics.

The VA-Model diagram shows levels that spiral up a triangle as reasoning advances. Visualization and analysis become closer to each other as one moves to more advance levels. The VAP-Model diagram is a tetrahedron, to accommodate the addition of a physical mode. The path between modes also spirals up with levels getting closer to each other, however, in the VA-diagram there is an orderly path that moves from visualization to analysis and back. In the VAP-diagram the path moves upward between three modes, but does so through a disorderly unpredictable path. This signifies that the transitions between visual, analytic and physical modes do not follow a specified sequence.

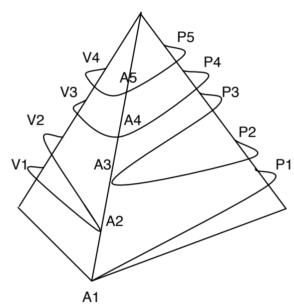

Figure 2: The VAP-Model Diagram

Students' transitions between the three modes of thinking, in the context of problem solving, are of particular interest because they inform how students use multiple modes of representation in conjunction with each other. The VAP-model contends that these modes inform each other, but what this looks like, when it happens or how such transitions can be fostered by instruction are not predicted by the model.

Instead of classifying individual students as predominantly preferring one mode of reasoning over another, I contend that the classification should be of students' claims and justifications and whether those are visual, analytic, or physical in nature. In other words I regard all students as harmonic reasoners to some extent, regardless of which representational mode is predominant in

their thinking. Within such a classification I place a special emphasis on back-and-forth transitions between modes of reasoning.

Categorizing Transitions

In the secondary grades translation tasks are often solely about the translation itself. In a calculus context, however, translation between modes is often part of the task and not the task itself. Some calculus tasks are stated in one representational mode and require an answer in another. For example, consider the following task: *"If f(0)=1, f '(0)=1, f(3)=7, f'(3)=-1 and f ''(3)=-1, sketch a possible graph of f(x)."* The task is stated in terms of the analytic mode, since the information about the function is given symbolically, and the answer is supposed to be provided in a graphical mode. Note that in order to complete the task a student is required to transition between representational modes. Solving the task requires moving from one edge of the VAP-diagram to another.

It can also be the case that translation between modes is not necessarily required in order to complete the task. However, spontaneous transitions between representational modes may occur anyway during students problem solving. For example if a student is given the following integral to solve $\int_{-3}^{3} x\sqrt{9-x^2}\,dx$, she may solve it using standard methods, such as u-substitution. The problem can, however, be solved by reasoning about the shape of the graph of $x\sqrt{9-x^2}$. The graph has a 180º rotational symmetry about the origin (odd function). Therefore every region above the x-axis has a corresponding region below the axis on the other side of the y-axis. Since the bounds of integration are symmetric with respect to the origin the integral evaluates to zero. Even though the problem is stated in symbolic/analytic terms and requires a symbolic/numerical answer the second solution makes extensive use of the graphical mode. If a student solves the task in this way, her thought process moves from one edge of the VAP-diagram to another, but this transition is not specifically required by the problem itself.

I refer to transitions between modes that are not required by the task, as *unprompted transitions*. Further, I refer to transitions that are part of the problem itself, that is, when a problem is stated in one mode and requires an answer stated in another, as *prompted transitions*. Note that prompted transitions are an attribute of a task and unprompted transitions are an attribute of a solution. So it is possible to have an unprompted transition occur within the context of a prompted transition problem.

Method

Data and Participants

This study followed a group of three average students as measured by their scores on the standardized Calculus Concept Readiness (CCR) test (Carlson, Madison &West, 2010). The group consisted of two males and one female, which were given the pseudonyms Carson, Brad and Ann. These students were observed over the course of a semester long technologically enriched calculus class taught at a large university in the southwestern United States. The class had approximately 70 students. The three students in this study worked together during in-class group work, which was recorded daily. Each of the three students also participated in three individual problem-solving interviews throughout the semester. The data in this article come from these interviews.

Analysis

All interview tasks were coded for which representation modes were prompted. Student work on these tasks was also coded for representational mode with special attention paid to when transitions occurred and how these transitions informed students' problem solving. The metonymy of many mathematical terms necessitated the use of a neutral code. The code was applied when it was unclear which mode of reasoning was being used.

Results

I begin by discussing the students' use of representations as a whole and how they relate to the representations in the task statements. Then I shift to discussing a particular student in detail.

Table 1 documents which transitions (if any) were explicitly required by each of the interview tasks and the transitions between representational modes students used in their solutions. Visual, analytic and physical are indicated with V, A and P, respectively.

Table 1: Prompted and Unprompted Transitions.

	Interview 1					**Interview 2**							**Interview 3**						
Task #	1	2	3[1]	4	5	1	2	3	4	5	6	7	1	2	3	4	5	6	7
Task Type	AP	AP	V	V	VP	AP	V	V	V	V	AP	VAP	AP	VP	AP	V	VAP	VP	VP
Ann	VAP	VAP	V	VA	VP	VAP	VP	VP	VAP	VAP	VAP	VAP	VAP	VP	VAP	VP	VAP	VP	VAP
Brad	VAP	VAP	V	V	VP	VAP	V	V	VA	VA	VAP	VAP	VAP	VP	VAP	V	VAP	VP	VP
Carson	VAP	VAP	VAP	VP	VP	VAP	VP	VP	VAP	VAP	VAP	VAP	VAP	VP	VAP	VP	VAP	VP	VP

Note that, in solutions to tasks that incorporate all three modes unprompted transitions are not possible. In table 1 two thirds of student solutions that could have shown an unprompted transition did. One important thing to notice from Table 1 is that most of the instances where students made an unprompted transition involved the addition of a visual or physical mode. The instances that incorporated unprompted transitions to the visual mode typically involved drawing a graph to aid with reasoning. Unprompted transitions to the physical mode typically involved reasoning about a graph as if it were describing the motion of some particle or vehicle, which was not part of the specified problem. A specific transition to the physical mode is explored in more detail below. The addition of an unprompted analytic component was rare but did occur in the data. This took the form of reasoning that a graph (or part of a graph) appeared similar to a known analytic function and then finding a related function analytically before translating back to the graphical mode. This behavior, which typifies analytic thinkers in Presmeg and her colleague's work, was fairly uncommon in this data set.

Table 1 documents what modes were used for particular questions but does not detail the specifics of how representational modes inform one another during the problem-solving process. The VAP-model contends that these transitions between modes are central to students' development. The next section details a particular set of transitions used to solve a graphing task.

[1] This is the task shown in Figure 3

Carson and the Derivative Sketching Task

Although each of the three interviews involved several tasks, only one task will be discussed due to space limitations. The derivative sketching task (Figure 3) is a graphing task that asks students to sketch the graph of a derivative given a particular original function. The task is similar to a task discussed by Aspinwall and Shaw (2002) that asked students to sketch the derivative of a continuous symmetric ‘saw-tooth’ graph that alternated between a slope of negative one and one. The graph in Figure 3, unlike Apsinwall and Shaw’s graph, alternates between several different slopes. These graphs have no simple translation into analytic notation. They therefore discourage an analytic approach. Aspinwall and Shaw observed that students they classified as analytic thinkers had difficulty with the saw-tooth task. As presented, the task below does not force any transitions between modes of representation since it can be solved using only graphical reasoning.

Sketch the graph of the derivative of the following function. Think aloud as you sketch your graph.

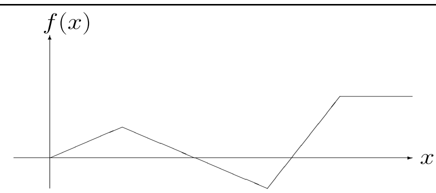

Figure 3: Derivative Sketching Task

Below is the transcript of Carson working on the derivative sketching task (Figure 3). This task is stated in graphic terms and requires a graphic solution. The transcript details a solution that does not stay solely within the confines of graphical thinking.

Carson [00:22:49]: Alright, so I know that the derivative is the slope and I took physics so I know that this is distance [writes a d under x axis] over, no that’s wrong this is time [crosses out d and writes t under the x axis]. This is time over distance, which is you speed. Speed is distance over time. So this is time and this is your speed [labels axis on derivative function] and so as your distance…I’m sorry… So this is constant so you know that velocity is constant. So your velocity is something like this [draws short horizontal line segment above x-axis] and then later when it hits this tip it’s at zero [marks a dot on the x-axis after previously drawn segment]. And then later when it’s decelerating. Ya this is a negative speed so the graph. And it's a straight line so you know it would be something like this [draws a horizontal line under the x-axis]. And then again at this point it’s zero [draws another dot on the x-axis after the second segment]. And then um accelerating. But this time it’s more this is steeper. So it would be higher because your velocity would be faster. Your speed would be faster. So it would be like this [draws a third horizontal line segment above the x-axis higher than the first]. And then again right here it’s zero [draws another dot on the x-axis]. And then now this one your distance isn’t changing. Since your distance isn’t changing. This equation [s=d/t] looks like zero over time. So the rest of the graph would look like this [draws a fourth line along the x-axis.]

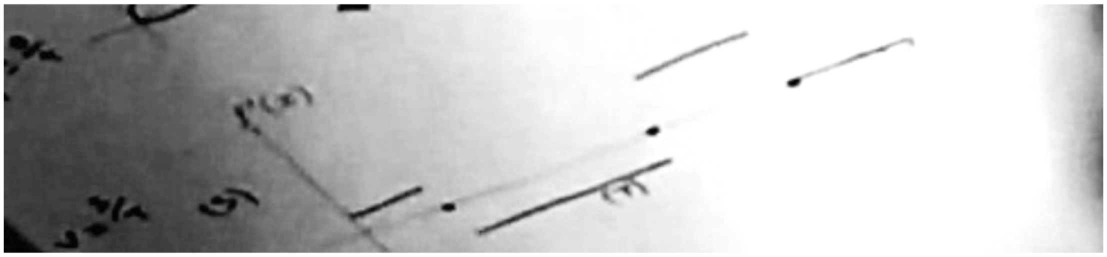

Figure 4: Carson's Solution to the Derivative Sketching Task.

In the above transcript Carson is presented with a question that makes no mention of a physical context, however, Carson attributes the function in the question to a function that describes a moving object. This transition to physical thinking is unprompted by the question. Carson does not simply shift into a physical mode and remain there. He continually moves back and forth between visual and physical modes. More specifically, he interprets a section of the given graph as corresponding to a physical motion, reasons about the velocity/speed of that motion and then translates that into a velocity graph. This cycle occurs several times throughout the transcript.

When dealing with the last segment of the function Carson switches to an analytic mode. He reasons that a non-changing position corresponds to 0/t=0, no change in distance over a non-zero change in time, before sketching the last segment of the derivative. So the above transcript shows unprompted transitions to both physical and analytic modes within a graphical problem. Relating this back to the VAP-diagram, Carson's reasoning continually alternates between the physical and analytic edges of the tetrahedron before moving to its analytic edge.

The connections that Carson makes between physical and graphical contexts led to some interesting artifacts. The given graph cannot represent the position graph of a physical object. Physical objects cannot instantaneously change directions and so it does not make sense to discuss the physical interpretation of what happens at those points. Carson, however, does not abandon the physical-graphical link. This leads him to conclude erroneously that there are zeros at the points where the graph instantaneously switches direction. This phenomenon is similar to one noted in Aspinwall, Shaw and Presmeg (1997), which they termed uncontrollable mental imagery. This is where visual images associated with students' graphical interpretations interfere with their analytic interpretations.

Further questioning revealed that Carson's translation into the physical mode only appears to affect his ability to deal with sudden transitions from increasing to decreasing or vice-versa. Consequently, his errors are limited to several discrete points. So, his use of the connection between graphical and physical modes appears to help him more than it hinders.

Discussion

In my view, calculus curricula that make non-trivial attempts to incorporate graphical and physical modes carry with them the implicit goal of fostering representational fluency. In other words, the goal of incorporating visual/graphical and kinesthetic/physical elements into a calculus course is not to expose students to separate modes of thinking, each of which targets a specific class of problems. Rather, the goal is to expose students to ways of approaching problems that can complement and elaborate each other. Simply exposing students to multiple representations does not ensure that they can translate between them. In order to better understand how to foster a rich back and forth relationship between modes of reasoning researchers need to understand what such transitions look like, how they evolve over time and

what kinds of tasks and teaching actions help foster them. This paper is a contribution to the first of these goals, illuminating what these transitions look like, both when they are prompted and unprompted. These transitions, at least in this particular data set are not uncommon. They occurred in two thirds of student solutions that could have shown an unprompted transition

If I were to stick solely to classifying general tendencies, Carson would be labeled a physical thinker because he used physical reasoning as part of his solution to every task. However, this labeling would have completely overlooked the rich unprompted transitions between modes of thinking that were integral to his problem solving processes.

These transitions between modes, in which one mode of reasoning informs another, are central to how the VAP-model views the development of analytic, visual and physical modes. More importantly, they shed light on what these transitions look like and may be used as a launching point for developing curricula and instruction that strengthens students' ability to make such transitions.

References

Aspinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable mental imagery: graphical connections between a function and its derivative. Educational Studies in Mathematics, 33, 301-317.

Aspinwall, L., & Shaw, K. (2002). Report on calculus: Two contrasting cases. Mathematics Teacher, 95(6), 434-439.

Bressoud, D. (2011). The calculus I instructor. Launchings, Washington, DC: Mathematical Association of America.

Carlson, M., Madison, B., & West, R (2010). The calculus concept readiness (CCR) instrument: Assessing student readiness for calculus. Preprint, arXiv.org.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. Educational Studies in Mathematics, 39(1/3), 111-129.

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting cases of calculus students' understanding of derivative graphs. Mathematical thinking and learning, 12(2), 152-176.

Hughes-Hallett, D.H., Gleason, A., Flath, D., Gordon, S., Lomen, D., Lovelock, D., McCallum, N., Osgood, B., Pasquale, A., Tecosky-Feldman, J., Thrash, J. & Thrash. K. (1994). Calculus. New York: John Wiley and Sons.

Kaput, J. J. (1987). Representation and mathematics. In C. Janvier (Ed.), Problems of representation in mathematics learning and problem solving (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kung, D.T. & Speer, N. (in press). What Could They Possibly Be Thinking!?! Understanding your college math students (MAA notes). Washington DC: Mathematics Association of America.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, 500-507.

Leinhardt, G., Zaslavsky, O., & Stein, M. M. (1990). Functions, graphs, and graphing: tasks, learning and teaching. Review of Educational Research, 60, 1–64.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. Cognition and Instruction, 16 (2), 119 – 172.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutierrez & P. Boero (Eds.), Handbook of research on the psychology of mathematics education: Past, present and future (pp. 205–235). Rotterdam, The Netherlands: Sense.

Vinner, S. (1989) The avoidance of visual considerations in calculus students. Focus on Learning Problems in Mathematics, 11(2), 149-156.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), Research in Collegiate Mathematics Education IV (Vol. 8, pp. 103–126). Providence, RI: American Mathematical Society.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating Visual and Analytic Strategies: A study of students' understanding of the group D_4. Journal for research in Mathematics Education, 435-457.

Zimmermann, W. (1991). Visual thinking in calculus. in W. Zimmermann and S. Cunningham (eds.), Visualization in Teaching and Learning Mathematics (pp. 127–137). Washington, DC: Mathematical Association of America.